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TWO APPROACHES TO THE STABILITY PROBLEM FOR PLASMA EQUILIBRIUM IN A CYLINDER†

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Using the example of the problem of the equilibrium of a plasma cylinder in a helical magnetic field, two approaches to the problem of its stability are proposed. The first approach uses the symmetry of the equilibrium configuration. Its model is constructed in terms of a boundary-value problem with the Grad–Shafranov equation. The equilibrium is said to be "diffusionally stable" if the solution of the problem can be obtained by iterative methods of the relaxation type. The stability is determined by the spectral property of the differential operator of the linearized equation. The other approach is the traditional linear theory of the MHD-stability of equilibrium configurations. After its schematic description, as it applies to a cylinder, it is shown that for both of the approaches considered the eigenvalues of the eigenvalue problems for helical harmonies of any small perturbation vanish simultaneously. This indicates that the stability boundaries in the range of the parameters of the problem are identical in both cases. © 2001 Elsevier Science Ltd. All rights reserved.

In numerous publications on the mathematical modelling of physical processes, relating to the problem of controlled thermonuclear fusion, considerable importance is attached to investigating the stability of the configurations of dense hot plasma, confined by a magnetic field ("magnetic traps"). The confinement time is assumed to be long on the scale of natural plasma processes, and hence configurations are considered which are in equilibrium. Two groups of problems are considered: (1) the geometry and physical parameters of the equilibrium configuration and (2) the stability of the configuration to perturbations which disturb its equilibrium. In each of these the investigations rest on a solution (usually numerical) of problems with the equations of magnetohydrodynamics (MHD).

The structure of plasma systems or their components often possess symmetry (plane, axial or helical). Hence, tow-dimensional mathematical models are widely used which can be simplified considerably compared with the general case. In two-dimensional equilibrium problems, the system of MHD-equations can be reduced to a single scalar Grad–Shafranov equation

$$\Delta\psi = g(\psi) \tag{0.1}$$

for the stream function of the magnetic field ψ [1–3]. Here Δ is the two-dimensional Laplace operator, considered in different systems of curvilinear coordinates depending on the type of symmetry. A feature of boundary-value problems with Eq. (0.1) is the fact that they can have non-unique solutions. Some of these solutions cannot be obtained by iterative methods of the "relaxation" type, i.e. by numerical integration of the evolution equation

$$\partial\psi/\partial t = \Delta\psi - g(\psi) \tag{0.2}$$

It is natural to assume them to be "unstable" in the sense indicated, while, depending on the nature of Eq. (0.2), the stability or instability is said to be "diffusional". The diagnostics and criteria of diffusional stability use a spectral analysis of the differential operator of the linearized equation (0.1) [4–6].

The questions of uniqueness and stability touched upon belong to the general theory of semilinear elliptic and parabolic equations [7]. In addition to two-dimensional problems of plasma statics [8], they also occur in problems of the theory of combustion (of any dimensionality) [9, 10], electrochemistry [11], and when modelling chemical processes [12]. The features of localized thermal regimes and regimes with peaking in the quasi-linear theory of heat conduction are of the same nature [13].

Diffusional stability must be distinguished from the traditionally considered MHD-stability of equilibrium plasma configurations with respect to hydrodynamic-type perturbations, including motion. Investigations of MHD-stability make up a second group of such problems and are a permanent subject matter in the scientific literature (see, for example, [14–19]). It is of interest to establish if there are any interrelations or hierarchy between both types of stability. The question is raised in [5, 20] and has no obvious answer. An investigation of the stability in the linear

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approximation in both cases involves a spectral analysis of differential operators, but of an extremely different nature. In the case of MHD three-dimensional perturbations are considered described by a system of equations, while the Grad–Shafranov equation and its linearized analogue are scalar and are limited by a two-dimensional analysis.

In this paper we consider this question using the simple example of a one-dimensional equilibrium configuration in a plasma cylinder in a helical magnetic field. This example is well known as a straightened model of toroidal systems (the tokamak and stellarator). Its MHD-stability has been the subject of numerous investigations (for example, [14–18, 21], while the diffusional stability has been considered in [20], in which a number of features and tendencies related to the right-hand side $g(\psi)$ of Eq. (0.1) are determined. A comparison of the two approaches to stability leads to the following results.

Arbitrary three-dimensional small MHD-perturbations can be represented by a Fourier series, composed of helical perturbations. Each of these has a corresponding two-dimensional helical diffusion perturbation, described by linearized equation (0.2). The eigenvalue problems connected with these perturbations are different, but they are identical in the case of eigenvalues equal to zero. Since the passage of the leading eigenvalue through zero give the limit of the stability in the range of the problem parameters, the result obtained indicates that these limits are identical for both forms of stability. Hence it follows that if the configuration is diffusionally stable for all values of the helix pitch, specified by the helical symmetry, it is stable for all Fourier harmonics of the MHD-perturbations. An investigation of diffusional stability is associated with simpler eigenvalue problems, and this is the practical importance of the result formulated.

Below, in Section 1, we give a schematic description of the two-dimensional model of the equilibrium in terms of the Grad–Shafranov equation and we introduce the idea of diffusional stability as it applies to a plasma cylinder with a helical field. In Section 2 we reproduce a scheme for investigating MHD-stability in the linear approximation, realized in the same cylinder. In Section 3 we compare both forms of stability and we obtain the main results of this paper.

1. TWO-DIMENSIONAL MODEL OF EQUILIBRIUM. "DIFFUSIONAL STABILITY"

The equilibrium configuration (i.e. which is at rest) of a dense plasma in a magnetic field is characterized by a distribution in space of three physical quantities: the pressure p, the magnetic field strength $\mathbf{H}$ and the electric current density $\mathbf{j}$. They are related by the MHD-equation of equilibrium

$$\nabla p = \mathbf{j} \times \mathbf{H} \tag{1.1}$$

and Maxwell's equations

$$\nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \mathbf{j} \tag{1.2}$$

The equations are given in dimensionless form: the units of measurement are made up of the quantities which participate in the setting of specific problems.

When there is symmetry (plane, axial or helical) the equations are two-dimensional and the mathematical apparatus of plasma statics is simplified considerably. Equations (1.2) enable us to describe the magnetic field and the current in the direction of two active coordinates using the scalar stream functions ψ and I. It follows from Eq. (1.1) that the functions p, ψ and I are pairwise dependent, i.e.

$$p = p(\psi), \quad I = I(\psi) \tag{1.3}$$

while the function ψ satisfies the scalar Grad–Shafranov equation (0.1), the specific form of which depends on the type of symmetry. For example, the helical symmetry discussed below assumes that the configuration depends solely on r and $\Theta = \varphi - \alpha z$, where r, φ and z are cylindrical coordinates, $\alpha = 2\pi/h$, and h is the pitch of the helix of the coordinate lines. In this case

$$rH_r = \frac{\partial\psi}{\partial\Theta}, \quad H_\Theta \equiv H_\varphi - \alpha r H_z = -\frac{\partial\psi}{\partial r}$$

$$rj_r = \frac{\partial I}{\partial\Theta}, \quad j_\Theta = -\frac{\partial I}{\partial r}; \quad I \equiv H_l = H_z + \alpha r H_\varphi$$

while the Grad–Shafranov equation has the form

$$\Delta^{**}\psi = g(\psi, r) \tag{1.4}$$

where

$$\Delta^{**}\psi \equiv \nabla\left(\frac{\nabla\psi}{\nu}\right) \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\nu}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\Theta^2}, \quad \nu \equiv 1 + \alpha^2 r^2$$

$$g(\psi, r) = -\frac{dp}{d\psi} + \frac{2\alpha}{\nu^2} I - \frac{I}{\nu}\frac{dI}{d\psi}$$

The right-hand side of Eq. (1.4) contains two arbitrary functions (1.3), which describe the pressure and electric current distribution between the magnetic surfaces $\psi(r, \Theta)$ = const. They must be specified, starting from any requirements and wishes to the configuration being considered or on the basis of a priori experimental information. The model of the equilibrium configuration is constructed by solving the boundary-value problem with Eq. (1.4) in the specified region with specified boundary conditions.

An extensive literature is devoted to the investigation of plasma configuration in terms of the model described (see, for example, [3, 16, 17]). Research on this subject, in which we have participated, is summarized in [4, 5]. In these, in particular, attention is drawn to the fact that the first boundary-value problem with an equation of the type (0.1) can have non-unique solutions. Some of these cannot, generally speaking, be obtained by iterative "time-relaxation" methods, i.e. using the solution of Eq. (0.2). For this reason they are said to be "diffusionally unstable".

We emphasize that stability or instability is not a consequence of the chosen method of solution, but an internal property of the problem, which is related to the spectrum of the differential operator of the linearized equation

$$L[u] \equiv -\Delta u + \frac{\partial g}{\partial\psi} u$$

with the condition $u = 0$ on the boundary of the region. That is, the criterion of the stability of the solution ψ of the boundary-value problem with Eq. (0.1) is the inequality $\lambda_1 > 0$, where λ_1 is the first (minimum) eigenvalue of the boundary-value problem with operator L

$$L[u] = \lambda u \tag{1.5}$$

In fact, the convergence of the relaxation method, i.e. the solution of Eq. (0.2) reaching the steady state (0.1), is determined by the behaviour of the error, which is subject to the equation

$$\partial u/\partial t + L[u] = 0 \tag{1.6}$$

and decreases or increases with time as $\exp(-\lambda_1 t)$. The inverse inequality $\lambda_1 < 0$ for any solution ψ of the problem with Eq. (0.1) denotes diffusional instability of this solution and, as a rule, the fact that it is non-unique.

In order to compare diffusional stability with MHD-stability we will consider the example of the one-dimensional equilibrium configuration in a plasma cylinder with a helical magnetic field. Equations (1.1) and (1.2) have the form

$$\frac{dp}{dr} + H_z\frac{dH_z}{dr} + \frac{H_\varphi}{r}\frac{dH_\varphi r}{dr} = 0 \tag{1.7}$$

$$H_r \equiv 0, \quad j_r \equiv 0; \quad j_\varphi = -\frac{dH_z}{dr}, \quad j_z = \frac{dH_\varphi r}{rdr} \tag{1.8}$$

Here, two functions, for example $H_z(r)$, $H_\varphi(r)$, must be known in the segment $0 \leqslant r \leqslant 1$, while the remaining ones are defined by Eqs (1.7) and (1.8).

This configuration can be regarded as a special case of the two-dimensional one, including it in the model with helical symmetry for an arbitrary value of the helical parameter α. We will introduce the stream function $\psi(r)$ by the equation

$$H_\Theta \equiv H_\varphi - \alpha r H_z = -d\psi / dr$$

It satisfies Eq. (1.4), in which

$$I \equiv H_l \equiv H_z + \alpha r H_\varphi; \quad \frac{dp}{d\psi} = \frac{dp}{dr}\left(\frac{d\psi}{dr}\right)^{-1}, \quad \frac{dI}{d\psi} = \frac{dI}{dr}\left(\frac{d\psi}{dr}\right)^{-1} \tag{1.9}$$

The two-dimensional helical perturbations of configuration (1.7), (1.8) obey Eq. (1.6), in which the coefficients of the operator L are independent of t and Θ, and hence they can be represented by the sum of a series of particular solutions of the form

$$u(t, r, \Theta) = e^{\lambda t} e^{im\Theta} u(r)$$

while

$$L[u] \equiv -\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\nu}\frac{du}{dr}\right) + \left(\frac{m^2}{r^2} + \frac{\partial g}{\partial \psi}\right) u = \lambda u \tag{1.10}$$

$$|u(0)| < \infty, \quad u(1) = 0$$

where $g(\psi, r)$ is the right-hand side of Eq. (1.4). Differentiating it in the same way as above (formulae (1.9)), we have

$$\nu H_\Theta \frac{\partial g}{\partial \psi} = \frac{d^2 H_\Theta}{dr^2} + \frac{1 - 3\alpha^2 r^2}{r\nu}\frac{dH_\Theta}{dr} - \frac{2\alpha^2 r^2 H_l}{r\nu H_\Theta}\frac{dH_l}{dr} - \frac{(1 + 6\alpha^2 r^2 - 3\alpha^4 r^4) H_\Theta + 8\alpha^3 r^3 H_l}{r^2 \nu^2} \tag{1.11}$$

The configuration is diffusionally stable if, for any pair of values of m and α, all $\lambda > 0$. Note that, for fixed α, the coefficient of u in Eq. (1.10) increases as m increases and, consequently, the eigenvalues λ increase (see, for example, [22]), improving the stability. Harmonics with number $m = 0$ (long-wave perturbations) are the least stable. Hence, the equilibrium configuration (1.7), (1.8) is stable to any two-dimensional helical perturbations if and only if the spectrum of the operator L is positive for $m = 0$ and all possible values of α.

2. A SCHEME FOR INVESTIGATING MHD-STABILITY

We will present a brief scheme for investigating MHD-stability in the linear approximation. Small perturbations of $\mathbf{V}_1, p_1$ and $\mathbf{H}_1$ of any equilibrium plasma configuration are described by the equations of magnetohydrodynamics, linearized in a neighbourhood of the state of rest (1.1), (1.2)

$$\rho \frac{\partial \mathbf{V}_1}{\partial t} = -\nabla p_1 + \mathbf{j}_1 \times \mathbf{H} + \mathbf{j} \times \mathbf{H}_1$$

$$\frac{\partial p_1}{\partial t} = -\gamma p \nabla \cdot \mathbf{V}_1 - \nabla p \cdot \mathbf{V}_1, \quad \frac{\partial \mathbf{H}_1}{\partial t} = \nabla \times (\mathbf{V}_1 \times \mathbf{H}) \tag{2.1}$$

Differentiating the first equation of (2.1) with respect to t and eliminating p_1 and $\mathbf{H}_1$, we obtain a vector equation solely in $\mathbf{V}_1$

$$\rho \frac{\partial^2 \mathbf{V}_1}{\partial t^2} = \nabla(\gamma p \nabla \cdot \mathbf{V}_1 + \nabla p \cdot \mathbf{V}_1) + \mathbf{j} \times (\nabla \times (\mathbf{V}_1 \times \mathbf{H})) + (\nabla \times \nabla \times (\mathbf{V}_1 \times \mathbf{H})) \times \mathbf{H} \equiv -\mathbf{K}[\mathbf{V}_1] \tag{2.2}$$

which is often used in the form

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\mathbf{K}[\boldsymbol{\xi}], \quad \boldsymbol{\xi} = \int \mathbf{V}_1 dt \tag{2.3}$$

as it applies to the small-displacement vector ξ. The latter obviously follows from system (2.1) or can be obtained from Eq. (2.2) by integration with respect to t. The form of (2.3) is more convenient when interpreting the mechanical meaning of stability: the right-hand side plays the role of a force while the quadratic form $\int \mathbf{K}[\xi]\cdot\xi d\mathbf{r}$ is the potential energy of the perturbation.

The coefficients of the linear operator $\mathbf{K}$ depend only on the equilibrium solution, and hence the problem with Eq. (2.3) and the boundary conditions, for example, $\xi_n = 0$ on the rigid boundary of the region, allows of the separation of variables

$$\boldsymbol{\xi}(t, \mathbf{r}) = e^{i\omega t}\boldsymbol{\xi}(\mathbf{r}) \tag{2.4}$$

and is reduced to a boundary-value problem with the equation

$$\rho\omega^2\boldsymbol{\xi} = \mathbf{K}[\boldsymbol{\xi}] \tag{2.5}$$

The operator $\mathbf{K}$ is selfadjoint over a wide range of natural boundary conditions, and hence its eigenvalues ω^2 are real. The equilibrium is stable if they are all positive; in this case the small displacements (2.4) do not increase with time. They are unstable if at least one $\omega^2 < 0$ exists.

Eigenvalue problems with Eqs (1.5) and (2.5) are not very common. The first is a scalar problem and is related to perturbations of dimensionality no higher than two, while the second is a vector problem and allows of three-dimensional perturbations.

As it applies to the one-dimensional equilibrium configuration in a cylinder (1.7), (1.8) considered above, Eq. (2.5) allows of a further separation of variables

$$\boldsymbol{\xi}(\mathbf{r}) = e^{im\varphi - ikz}\boldsymbol{\xi}(r) = e^{im\Theta}\boldsymbol{\xi}(r)$$

where

$$\Theta = \varphi - \alpha z, \quad \alpha = k/m, \quad \boldsymbol{\xi}(r) = (\xi_r, i\xi_\varphi, i\xi_z)$$

Putting

$$\xi_\Theta = \xi_\varphi - \alpha r\xi_z, \quad \eta = \frac{m}{r}\xi_\Theta, \quad \zeta = H_z\xi_\varphi - H_\varphi\xi_z$$
$$H_\Theta = H_\varphi - \alpha r H_z, \quad H_l = H_z + \alpha r H_\varphi$$

we obtain from Eq. (2.5)

$$\begin{aligned}
\rho\omega^2\xi_r &= K_r(\boldsymbol{\xi}) \equiv K_0[\xi_r] + \left(\frac{m}{r}H_\Theta\right)^2\xi_r + \frac{d}{dr}(\gamma p\eta) + \frac{d}{dr}\left(\frac{m}{r}H_l\zeta\right) + \frac{2kH_\varphi}{r}\zeta \\
\rho\omega^2\eta &= K_\eta(\boldsymbol{\xi}) \equiv -\frac{m^2}{r^2}(\nu\gamma p + H_l^2)\frac{d\xi_r r}{r dr} + 2kH_\varphi\frac{m}{r^2}H_l\xi_r + \frac{m^2\nu}{r^2}\left(\gamma p\eta + \frac{m}{r}H_l\zeta\right) \\
\rho\omega^2\zeta &= K_\zeta(\boldsymbol{\xi}) \equiv -\frac{mH_l}{r}(\gamma p + H^2)\frac{d\xi_r r}{r dr} + 2kH^2\frac{H_\varphi}{r}\xi_r + \frac{m}{r}H_l\gamma p\eta + \frac{m^2}{r^2}H^2\nu\zeta
\end{aligned} \tag{2.6}$$

where

$$K_0[\xi_r] \equiv -\frac{d}{dr}\left(\rho C_s^2\frac{d\xi_r r}{r dr}\right) + \frac{d}{dr}\left(\frac{H_\varphi}{r}\right)^2\xi_r r$$
$$\rho C_s^2 = \gamma p + H^2; \quad H^2 = H_z^2 + H_\varphi^2$$

We resolve the last two equations of (2.6) for η and ζ and we substitute the result into the first equation of (2.6). The vector boundary-value problem with Eq. (2.5) is then converted into a scalar problem [15]

$$-\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\nu}F\frac{d\xi_r r}{dr}\right) + G\xi_r r = 0; \quad |\xi_r(0)| < \infty, \quad \xi_r(1) = 0 \tag{2.7}$$

where

$$r^2F(r,\omega)=\left(1-\frac{S^2}{D}\right)(H_\Theta^2-S)$$

$$r^2G(r,\omega)=\frac{m^2}{r^2}(H_\Theta^2-S)-\frac{4\alpha^2H_\varphi^2}{D}(\gamma pH_\Theta^2-H^2S)+r\frac{d}{dr}\left(\frac{H_\varphi}{\nu r^2}(2H_\Theta-\nu H_\varphi)+2\alpha rH_l\frac{S^2}{D}\right)$$

$$S=\frac{\rho\omega^2r^2}{m^2},\quad D=\nu\gamma pH_\Theta^2-\nu\rho C_s^2r^2S+S^2$$

Its specific feature is the fact that the eigenvalue ω^2 occurs non-linearly in the coefficients F and G. One of the possibilities of investigating the MHD-stability of plasma cylinder is to choose ω^2 by the "shooting" method in a series of numerical solutions of Eq. (2.7) [18].

3. THE COMMON LIMIT OF STABILITY

Eigenvalue problems (1.10) and (2.7), which differ considerably from one another, both relate to harmonics of the perturbations (diffusional and MHD) with fixed values of the number m and of the helical parameter α. The conditions for which the least eigenvalues λ and ω^2 vanish correspond to the boundaries of both types of stability of these harmonics.

Equation (2.7) with $\omega^2 = 0$ can be simplified considerably. When $m \neq 0$ it has the form

$$-\frac{1}{r}\frac{d}{dr}\left(\frac{H_\Theta^2}{\nu r}\frac{d\xi_r r}{dr}\right)+\left[\left(\frac{mH_\Theta}{r^2}\right)^2-\frac{1}{\nu}\left(\frac{2\alpha H_\varphi}{r}\right)^2-\frac{1}{r}\frac{d}{dr}\left(\frac{H_\varphi^2}{r^2}-\frac{2H_\varphi H_\Theta}{\nu r^2}\right)\right]\xi_r r=0 \qquad (3.1)$$

and when $m = 0, k \neq 0$ $(\alpha = \infty)$

$$-r\frac{d}{dr}\left(\frac{H_z^2}{r}\frac{d\xi_r r}{dr}\right)+\left[(kH_z)^2-2\frac{H_\varphi}{r}\frac{dH_\varphi}{dr}-2\frac{H_\varphi^2}{r^2}\right]\xi_r r=0 \qquad (3.2)$$

Putting $u = H_\Theta\xi_r$, we obtain from (3.1)

$$-\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\nu}\frac{du}{dr}\right)+\left(\frac{m^2}{r^2}+Q\right)u=0,\quad Q=\frac{\partial g}{\partial\psi} \qquad (3.3)$$

Here Q is identical with the expression $\partial g/\partial\psi$, defined by (1.11). Hence, Eq. (3.3) is identical with Eq. (1.10) when $\lambda = 0$. In the limiting case (3.2) the replacement $v = H_z\xi_r r$ leads to the equation

$$-r\frac{d}{dr}\left(\frac{1}{r}\frac{dv}{dr}\right)+(k^2+Q^*)v=0 \qquad (3.4)$$

where

$$Q^*=\frac{1}{H_z}\frac{d^2H_z}{dr^2}-\frac{1}{rH_z}\frac{dH_z}{dr}-\frac{2H_\varphi}{rH_z^2}\frac{dH_\varphi}{dr}-\frac{2H_\varphi^2}{r^2H_z^2}$$

which is identical with the Grad–Shafranov equation, linearized on equilibrium (1.7), (1.8), in its axisymmetrical version

$$\Delta^*\psi\equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2}=-r^2\frac{dp}{d\psi}-I\frac{dI}{d\psi}\equiv g^*(\psi,r)$$

Here

$$\frac{d\psi}{dr}=rH_z,\quad I=rH_\varphi,\quad Q^*=\frac{\partial g^*}{\partial\psi}$$

Finally, the limiting case $m = 0$, $\alpha \to \infty$, $H_z \equiv 0$ corresponds to a Z-pinch, the stability of which has been exhaustively investigated (see, for example, [14, 17]).

Hence, the eigenfunctions of both problems (1.10) and (2.7), corresponding to the zeroth eigenvalue, satisfy the same equation (3.3) with boundary conditions $|u(0)| < \infty$, $u(1) = 0$. This indicates that the boundaries of stability in the range of the parameters of the problem are identical in both cases considered for each of the perturbation harmonics.

The case of completely one-dimensional perturbations of the cylinder, $m = 0$, $k = 0$, $\alpha = 0$, is an exception. Here the MHD-stability is unconditional, at least, for values of the adiabatic index $\gamma \leqslant 2$. In fact, in this case it follows from Eqs (2.6) that $\xi_\varphi \equiv \xi_z \equiv 0$, and $\xi = \xi_r$ satisfies the equation

$$\rho\omega^2\xi = K_0[\xi]$$

The stability follows from the fact that the operator K_0 is positive-definite. It can be conveniently rewritten in the form

$$K_0[\xi] \equiv -\frac{1}{r}\frac{d}{dr}\left(r\rho C_s^2\frac{d\xi}{dr}\right) + q\xi, \quad q \equiv r\frac{d}{dr}\left(\frac{H_\varphi}{r}\right)^2 - \frac{d}{dr}\left(\frac{\rho C_s^2}{r}\right)$$

Using the equilibrium equation (1.7), we obtain

$$q = \frac{2-\gamma}{r}\frac{dp}{dr} + \frac{1}{r}\frac{dH_\varphi^2}{dr} + \frac{\rho C_s^2}{2}$$

$$\rho C_s^2 = \gamma p + H_z^2 + H_\varphi^2 = 2(\gamma - 1)p + H_z^2 + (2-\gamma)p + H_\varphi^2$$

Hence we have

$$\int_0^1 K_0(\xi)\xi r dr = \int_0^1 \left\{[2(\gamma-1)p + H_z^2]\left[\left(\frac{d\xi}{dr}\right)^2 + \frac{\xi^2}{r^2}\right] + [(2-\gamma)p + H_\varphi^2]\left(\frac{d\xi}{dr} - \frac{\xi}{r}\right)^2\right\} r dr \geqslant 0$$

for any ξ and for $\gamma \leqslant 2$ ("the energy principle" or the "energy integral" [14, 17]).

The diffusional stability of a column to one-dimensional perturbations, as previously, is non-trivial (see the examples in [20]).

The above comparison enables us to draw the following conclusion. In order to ascertain the fact of the MHD-stability of an equilibrium plasma configuration in a cylinder with a helical magnetic field, it is sufficient to establish the diffusional stability of all the helical perturbation harmonics. This means that the spectra of problems (1.10) when $m = 0$ and for any values of α must be positive. It at least one harmonic is diffusionally unstable, the corresponding boundary will be surmounted in the configuration parameters, and that harmonic will be unstable in the traditional hydrodynamic sense.

In conclusion, we draw attention to the fact the eigenvalue problem (1.10), in addition to the obvious singularity when $r = 0$, has one other, namely, when $r = r_c$, where $H_\Theta(r_c) = 0$. This corresponds to the equality.

$$\mu(r) \equiv \frac{H_\varphi}{rH_z} = \alpha \tag{3.5}$$

i.e. to the coincidence of the "twisting angle" $\mu(r)$ of the equilibrium magnetic lines of force and the helical parameter α. In other words, the line of force in this case coincides with the coordinate line, along which the perturbations of the given harmonic are assumed to be constant (helical symmetry). This coincidence is called "resonance", and the cylindrical surface $r = r_c$ is called the resonance surface [16]. It is well known that perturbations on a resonance surface are the most dangerous from the point of view of stability; examples given in [20] showed instability precisely in the region of resonance. Equation (3.3), which participates in both of the approaches to the problem of stability discussed above, correctly reflects this fact.

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